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Interval criteria for oscillation of second-order nonlinear differential equations

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Abstract

By employing the generalized Riccati technique and the integral averaging technique, new interval oscillation criteria are established for second-order nonlinear differential equations.

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1. Introduction

We are concerned here with the oscillatory behavior of second-order nonlinear differential equations of the form

$$(p(t)y'(t))' + f(t, y(t), y'(t)) = 0, \qquad t \geqslant t_0, \tag{1.1}$$

where $t_0 \geqslant 0$, $p \in C([t_0, \infty), (0, \infty))$, $f \in C([t_0, \infty) \times \mathbf{R}^2, \mathbf{R})$.

We shall assume that functions $p(t)$ and $f(t, y, z)$ are sufficiently smooth so that Eq. (1.1) always has solutions that are continuable throughout $[t_0, \infty)$. Such a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Finally, Eq. (1.1) is called oscillatory if all its solutions are oscillatory.

The oscillation problem for various particular cases of (1.1) such as the forced linear differential equation

$$(p(t)y'(t))' + q(t)y(t) = e(t), \tag{1.2}$$

the forced nonlinear differential equations

$$y''(t) + q(t)|y(t)|^{\gamma} \operatorname{sgn} y(t) = e(t), \qquad \gamma > 1, \tag{1.3}$$

$$(p(t)y'(t))' + q(t)f(y(t)) = e(t) \tag{1.4}$$

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and

$$(p(t)y'(t))' + q(t)f(y(t))g(y'(t)) = e(t) \tag{1.5}$$

has been studied extensively in recent years, e.g., see [1–7] and references therein.

Motivated by the ideas of El-Sayed [1], Nasr [3] and Wong [6], in the present paper by employing the generalized Riccati technique and the integral averaging technique, we shall establish several new interval criteria for oscillation of Eq. (1.1), that is, criteria given by the behavior of Eq. (1.1) (or p and f) only on a sequence of subintervals of $[t_0, \infty)$. Our results extend, improve and unify a number of existing results and handle some cases not covered by known criteria. Finally, two interesting examples are also included to show the versatility of our results.

2. Main results

In the sequel, for any $[a, b] \subset [t_0, \infty)$, we use the notation

$$D(a, b) := \{u \in C^1[a, b] : u(t) \not\equiv 0,\ \ u(a) = u(b) = 0\}.$$

Lemma 2.1. *Assume that y is a solution of Eq.* (1.1) *such that $y(t) \neq 0$ on some interval $[a, b] \subset [t_0, \infty)$. Let*

$$v(t) = -\frac{p(t)y'(t)}{y(t)} \tag{2.1}$$

on $[a, b]$. Then for any $u \in D(a, b)$, we have

$$\int_a^b \left[u^2(t)\frac{f(t, y(t), y'(t))}{y(t)} - p(t)u'^2(t)\right] \mathrm{d}t < 0. \tag{2.2}$$

Proof. Differentiating (2.1) and making use of (1.1), we see that $v(t)$ solves the Riccati equation

$$v'(t) = \frac{f(t, y(t), y'(t))}{y(t)} + \frac{v^2(t)}{p(t)}. \tag{2.3}$$

On multiplying (2.3) by u^2 for any $u \in D(a, b)$, integrating over $[a, b]$ and using integration by parts, we have

$$\begin{aligned}\int_a^b u^2(t)\,\frac{f(t, y(t), y'(t))}{y(t)}\,\mathrm{d}t &= \int_a^b u^2(t)\left[v'(t) - \frac{v^2(t)}{p(t)}\right] \mathrm{d}t \\ &= \int_a^b \left[-2u(t)u'(t)v(t) - u^2(t)\,\frac{v^2(t)}{p(t)}\right] \mathrm{d}t \\ &= -\int_a^b \left[\sqrt{p(t)}u'(t) + \frac{u(t)v(t)}{\sqrt{p(t)}}\right]^2 \mathrm{d}t + \int_a^b p(t)u'^2(t)\,\mathrm{d}t.\end{aligned}$$

It follows that

$$\int_a^b \left[u^2(t)\,\frac{f(t, y(t), y'(t))}{y(t)} - p(t)u'^2(t)\right] \mathrm{d}t = -\int_a^b \left[\sqrt{p(t)}u'(t) + \frac{u(t)v(t)}{\sqrt{p(t)}}\right]^2 \mathrm{d}t \leqslant 0. \tag{2.4}$$

Now, we claim that the equality in the last inequality of (2.4) fails to hold. Otherwise, we get that

$$u'(t) - \frac{u(t)y'(t)}{y(t)} = y(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u(t)}{y(t)}\right) = 0$$

on $[a, b]$. Also $y(t) \neq 0$ on $[a, b]$, so it follows that $u(t) = cy(t)$ for some constant c. Because $u(t) \in D(a, b)$ and $u \not\equiv 0$, this is incompatible with the fact that $y(t) \neq 0$ on $[a, b]$. This contradiction proves that the claim holds. □

The following theorem is an immediate result from Lemma 2.1.

Theorem 2.2. *If for some interval* $[a, b] \subset [t_0, \infty)$, *there exists* $u \in D(a, b)$ *such that for any* $y \in C^1([t_0, \infty), \mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a, b]$,

$$\int_a^b \left[u^2(t) \frac{f(t, y(t), y'(t))}{y(t)} - p(t)u'^2(t) \right] \mathrm{d}t \geqslant 0, \tag{2.5}$$

then every solution of Eq. (1.1) *has at least one zero in* $[a, b]$.

Proof. Suppose to the contrary that there exists a solution $y(t)$ of Eq. (1.1) such that $y(t) \neq 0$ on $[a, b]$. Then from the assumption, we see that there exists $u \in D(a, b)$ such that (2.5) holds for the given solution $y(t)$. However, from Lemma 2.1 we get that (2.2) holds, which contradicts assumption (2.5). □

From Theorem 2.2, we have the following oscillation criterion.

Theorem 2.3. *If for any* $T \geqslant t_0$, *there exist* $T \leqslant a < b$ *and* $u \in D(a, b)$ *such that for any* $y \in C^1([t_0, \infty), \mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a, b]$, (2.5) *holds, then Eq.* (1.1) *is oscillatory.*

Proof. Pick up a sequence $\{T_j\} \subset [t_0, \infty)$ such that $T_j \to \infty$ as $j \to \infty$. By the assumption, for each $j \in \mathbf{N}$, there exist $a_j, b_j \in [t_0, \infty)$ such that $T_j \leqslant a_j < b_j$, and (2.5) holds, where a, b are replaced by a_j, b_j, respectively. From Theorem 2.2, every solution $y(t)$ of Eq. (1.1) has at least one zero, $t_j \in [a_j, b_j]$. Noting that $t_j \geqslant a_j \geqslant T_j$, $j \in \mathbf{N}$, we see that every solution has arbitrarily large zeros. Thus, Eq. (1.1) is oscillatory. □

With appropriate choices of the function $f(t, y, z)$, from Theorem 2.3 we can derive a number of oscillation criteria, which extend, improve and unify a number of existing results. In fact, if $yf(t, y, z) \geqslant q(t)y^2 - e(t)y$ for all $(t, y, z) \in [t_0, \infty) \times \mathbf{R}^2$, we have the following corollary.

Corollary 2.4. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a_1 < b_1 \leqslant a_2 < b_2$ *and* $q, e \in C([t_0, \infty), \mathbf{R})$ *such that* $e(t)$ *has different signs on* $[a_1, b_1]$ *and* $[a_2, b_2]$, *and*

$$yf(t, y, z) \geqslant q(t)y^2 - e(t)y \quad \text{for all } (t, y, z) \in [t_0, \infty) \times \mathbf{R}^2.$$

If there exists $u \in D(a_i, b_i)$ *such that*

$$\int_{a_i}^{b_i} (qu^2 - pu'^2)\, \mathrm{d}t \geqslant 0,$$

for $i = 1, 2$, *then Eq.* (1.1) *is oscillatory.*

To establish the next corollary, we need the well-known Hölder inequality (see [8]): if $1 < \lambda < \infty$, $1 < \mu < \infty$, $(1/\lambda) + 1/\mu = 1$, X and Y are nonnegative, then $(1/\lambda)X + (1/\mu)Y \geqslant X^{1/\lambda}Y^{1/\mu}$.

Corollary 2.5. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a_1 < b_1 \leqslant a_2 < b_2$, $v > 1$ *and* $q, e \in C([t_0, \infty), \mathbf{R})$ *such that* $q(t) \geqslant 0$ *on* $[a_1, b_1] \cup [a_2, b_2]$, $e(t)$ *has different signs on* $[a_1, b_1]$ *and* $[a_2, b_2]$, *and* $f(t, y, z)/y \geqslant q(t)|y|^{v-1} - e(t)/y$ *for all* $t \in [a_1, b_1] \cup [a_2, b_2]$, $y \neq 0$, $z \in \mathbf{R}$. *If there exists* $u \in D(a_i, b_i)$ *such that*

$$v(v-1)^{(1-v)/v} \int_{a_i}^{b_i} u^2(t) q^{1/v}(t) |e(t)|^{(v-1)/v}\, \mathrm{d}t \geqslant \int_{a_i}^{b_i} p(t)u'^2(t)\, \mathrm{d}t,$$

for $i = 1, 2$, *then Eq.* (1.1) *is oscillatory.*

The next theorem is an extension of Theorem 2.3. For the sake of convenience, we give the following definition.

Definition 2.6. We say that a function $g(t)$ belongs to a function class $\mathscr{G}$, denoted by $g \in \mathscr{G}$, if $g \in C^1(\mathbf{R}, \mathbf{R})$ with $yg(y) > 0$ and $g'(y) > 0$ for all $y \neq 0$. For a positive constant K, the subset $\mathscr{G}_K$ of $\mathscr{G}$ is denoted by

$$\mathscr{G}_K := \{g \in \mathscr{G} | g'(y) \geqslant K \text{ for all } y \neq 0\}.$$

Theorem 2.7. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a < b$, $u \in D(a,b)$ *and* $g \in \mathscr{G}$ *such that for any* $y \in C^1([t_0,\infty),\mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a,b]$,

$$\int_a^b \left[u^2(t)\frac{f(t,y(t),y'(t))}{g(y(t))} - \frac{p(t)}{g'(y(t))}u'^2(t)\right] \mathrm{d}t \geqslant 0. \tag{2.6}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.1), say $y(t) \neq 0$ when $t \geqslant T_0$ for some T_0 depending on the solution $y(t)$. From the hypothesis, it follows that for any $T \geqslant T_0$, there exist $T \leqslant a < b$, $u \in D(a,b)$ and $g \in \mathscr{G}$ such that (2.6) holds for the given solution $y(t)$.

Now, we define for $t \geqslant T_0$

$$v(t) = -\frac{p(t)y'(t)}{g(y(t))}, \tag{2.7}$$

then $v(t)$ solves the Riccati equation

$$v'(t) = \frac{f(t,y(t),y'(t))}{g(y(t))} + \frac{g'(y(t))}{p(t)}v^2(t). \tag{2.8}$$

Multiplying (2.8) by u^2 and integrating over $[a,b]$, we have

$$\begin{aligned}
&\int_a^b u^2(t)\frac{f(t,y(t),y'(t))}{g(y(t))}\,\mathrm{d}t \\
&\quad = \int_a^b u^2(t)\left[v'(t) - \frac{g'(y(t))}{p(t)}v^2(t)\right]\mathrm{d}t \\
&\quad = \int_a^b \left[-2u(t)u'(t)v(t) - u^2(t)\frac{g'(y(t))}{p(t)}v^2(t)\right]\mathrm{d}t \\
&\quad = -\int_a^b \left[\sqrt{\frac{p(t)}{g'(y(t))}}u'(t) + u(t)v(t)\sqrt{\frac{g'(y(t))}{p(t)}}\right]^2\mathrm{d}t \\
&\qquad + \int_a^b \frac{p(t)}{g'(y(t))}u'^2(t)\,\mathrm{d}t.
\end{aligned}$$

This and (2.6) yield

$$u'(t) - \frac{u(t)g'(y(t))y'(t)}{g(y(t))} = g(y(t))\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u(t)}{g(y(t))}\right) = 0$$

on $[a,b]$. Also $y(t) \neq 0$ for $t \geqslant T_0$, so it follows that $g(y(t)) \neq 0$ and then $u(t) = cg(y(t))$ for some constant c. Because $u(t) \in D(a,b)$ and $u \not\equiv 0$, this is incompatible with the fact that $g(y(t)) \neq 0$ for $t \geqslant T_0$. This contradiction completes the proof. □

If $g'(y) \geqslant K > 0$ for all $y \neq 0$, we have the following corollary.

Corollary 2.8. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a < b$, $u \in D(a,b)$ *and* $g \in \mathscr{G}_K$ *such that for any* $y \in C^1([t_0,\infty),\mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a,b]$,

$$\int_a^b \left[u^2(t)\frac{f(t,y(t),y'(t))}{g(y(t))} - \frac{p(t)}{K}u'^2(t)\right]\mathrm{d}t \geqslant 0.$$

Then Eq. (1.1) *is oscillatory.*

By employing more general Riccati substitutions than (2.1) and (2.7) (see the following (2.10) and (2.13)), we have the following criteria.

Theorem 2.9. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a < b$, $u \in D(a,b)$, $\alpha \in C^1([t_0,\infty),(0,\infty))$, $\rho \in C([t_0,\infty),\mathbf{R})$ *and* $g \in \mathcal{G}$ *such that* $p(t)\rho(t) \in C^1([t_0,\infty),\mathbf{R})$ *and for any* $y \in C^1([t_0,\infty),\mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a,b]$,

$$\int_a^b \left\{ u^2(t)\Phi_1(t) - \frac{\alpha(t)p(t)}{g'(y(t))}\left[u'(t) + u(t)\left(\frac{\alpha'(t)}{2\alpha(t)} + \rho(t)g'(y(t))\right)\right]^2 \right\} \mathrm{d}t > 0, \tag{2.9}$$

where

$$\Phi_1(t) = \alpha(t)\left\{ \frac{f(t, y(t), y'(t))}{g(y(t))} + p(t)\rho^2(t)g'(y(t)) - [p(t)\rho(t)]' \right\}.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.1), say $y(t) \neq 0$ when $t \geqslant T_0$ for some T_0 depending on the solution $y(t)$. From the hypothesis, it follows that for any $T \geqslant T_0$, there exist $T \leqslant a < b$, $u \in D(a,b)$, $\alpha \in C^1([t_0,\infty),(0,\infty))$, $\rho \in C([t_0,\infty),\mathbf{R})$ and $g \in \mathcal{G}$ such that (2.9) holds for the given solution $y(t)$.

Now, we define for $t \geqslant T_0$

$$v(t) = -\alpha(t)p(t)\left\{ \frac{y'(t)}{g(y(t))} + \rho(t) \right\}, \tag{2.10}$$

then $v(t)$ solves the Riccati equation

$$v'(t) = \Phi_1(t) + \left(\frac{\alpha'(t)}{\alpha(t)} + 2\rho(t)g'(y(t))\right)v(t) + \frac{g'(y(t))}{\alpha(t)p(t)}v^2(t) \tag{2.11}$$

with $\Phi_1(t)$ given above.

Multiplying (2.11) by u^2 and integrating over $[a,b]$, we have

$$\begin{aligned}
&\int_a^b u^2(t)\Phi_1(t)\,\mathrm{d}t \\
&= \int_a^b u^2(t)\left[v'(t) - \left(\frac{\alpha'(t)}{\alpha(t)} + 2\rho(t)g'(y(t))\right)v(t) - \frac{g'(y(t))}{\alpha(t)p(t)}v^2(t)\right]\mathrm{d}t \\
&= \int_a^b \left\{-\left[2u(t)u'(t) + u^2(t)\left(\frac{\alpha'(t)}{\alpha(t)} + 2\rho(t)g'(y(t))\right)\right]v(t) - u^2(t)\frac{g'(y(t))}{\alpha(t)p(t)}v^2(t)\right\}\mathrm{d}t \\
&= -\int_a^b \left\{\sqrt{\frac{\alpha(t)p(t)}{g'(y(t))}}\left[u'(t) + u(t)\left(\frac{\alpha'(t)}{2\alpha(t)} + \rho(t)g'(y(t))\right)\right] + u(t)v(t)\sqrt{\frac{g'(y(t))}{\alpha(t)p(t)}}\right\}^2\mathrm{d}t \\
&\quad + \int_a^b \frac{\alpha(t)p(t)}{g'(y(t))}\left[u'(t) + u(t)\left(\frac{\alpha'(t)}{2\alpha(t)} + \rho(t)g'(y(t))\right)\right]^2\mathrm{d}t \\
&\leqslant \int_a^b \frac{\alpha(t)p(t)}{g'(y(t))}\left[u'(t) + u(t)\left(\frac{\alpha'(t)}{2\alpha(t)} + \rho(t)g'(y(t))\right)\right]^2\mathrm{d}t,
\end{aligned}$$

which contradicts assumption (2.9). □

If $g'(y) \geqslant K > 0$ for all $y \neq 0$ and $\rho(t) \equiv 0$ for $t \geqslant t_0$ in Theorem 2.9, we have the following corollary.

Corollary 2.10. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a < b$, $u \in D(a,b)$, $\alpha \in C^1([t_0,\infty),(0,\infty))$ *and* $g \in \mathcal{G}_K$ *such that for any* $y \in C^1([t_0,\infty),\mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a,b]$,

$$\int_a^b \alpha(t)\left[u^2(t)\frac{f(t,y(t),y'(t))}{g(y(t))} - \frac{p(t)}{K}\left(u'(t)+u(t)\frac{\alpha'(t)}{2\alpha(t)}\right)^2\right]\mathrm{d}t > 0.$$

Then Eq. (1.1) *is oscillatory.*

If $g'(y) \geqslant K > 0$ for all $y \neq 0$ and $\rho(t) = -(1/2K)(\alpha'(t)/\alpha(t))$ for $t \geqslant t_0$, we have the following modification of Theorem 2.9.

Theorem 2.11. *Suppose that for any* $T \geqslant t_0$, *there exist* $T \leqslant a < b$, $u \in D(a,b)$, $\alpha \in C^1([t_0,\infty),(0,\infty))$ *and* $g \in \mathcal{G}_K$ *such that* $p(t)\alpha'(t) \in C^1([t_0,\infty),\mathbf{R})$ *and for any* $y \in C^1([t_0,\infty),\mathbf{R})$ *with* $y(t) \neq 0$ *on* $[a,b]$,

$$\int_a^b \left[u^2(t)\Phi_2(t) - \frac{1}{K}\alpha(t)p(t)u'^2(t)\right]\mathrm{d}t > 0, \tag{2.12}$$

where

$$\Phi_2(t) = \alpha(t)\frac{f(t,y(t),y'(t))}{g(y(t))} - \frac{p(t)\alpha'^2(t)}{4K\alpha(t)} + \frac{1}{2K}[p(t)\alpha'(t)]'.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.1), say $y(t) \neq 0$ when $t \geqslant T_0$ for some T_0 depending on the solution $y(t)$. From the hypothesis, it follows that for any $T \geqslant T_0$, there exist $T \leqslant a < b$, $u \in D(a,b)$, $\alpha \in C^1([t_0,\infty),(0,\infty))$ and $g \in \mathcal{G}_K$ such that (2.12) holds for the given solution $y(t)$.

Now, we define for $t \geqslant T_0$

$$v(t) = -\alpha(t)p(t)\left\{\frac{y'(t)}{g(y(t))} - \frac{1}{2K}\frac{\alpha'(t)}{\alpha(t)}\right\}, \tag{2.13}$$

then

$$\begin{aligned} v'(t) &= \frac{\alpha'(t)}{\alpha(t)}v(t) - \alpha(t)\left\{\frac{(p(t)y'(t))'}{g(y(t))} - p(t)g'(y(t))\frac{y'^2(t)}{g^2(y(t))} - \frac{1}{2K}\left[p(t)\frac{\alpha'(t)}{\alpha(t)}\right]'\right\} \\ &\geqslant \frac{\alpha'(t)}{\alpha(t)}v(t) + \alpha(t)\frac{f(t,y(t),y'(t))}{g(y(t))} + K\alpha(t)p(t)\left(-\frac{v(t)}{\alpha(t)p(t)} + \frac{1}{2K}\frac{\alpha'(t)}{\alpha(t)}\right)^2 + \frac{\alpha(t)}{2K}\left[p(t)\frac{\alpha'(t)}{\alpha(t)}\right]' \\ &= \Phi_2(t) + \frac{K}{\alpha(t)p(t)}v^2(t) \end{aligned}$$

with $\Phi_2(t)$ given above. The rest of the proof is similar to that of Theorem 2.9 and hence omitted. □

3. Examples

In this section, we will show the application of our oscillation criteria by two examples. We will see that the equations in the examples are different from Eqs. (1.2)–(1.5) and oscillatory based on the results in Section 2, though the oscillations cannot be demonstrated by the results of [1–7] and most other known criteria.

We first give an example to show Corollary 2.4.

Example 3.1. Consider the forced second-order differential equation

$$\left[\sqrt{t+1}\,y'\right]' + q(t)y(1+\alpha y^2+\beta y'^2) = \sin\sqrt{t+1}, \tag{3.1}$$

where $t \geqslant 0$, $q(t) \in C([0,\infty),(0,\infty))$ and $q(t) \geqslant 1/4\sqrt{t+1}$; α and β are nonnegative constants.

In Eq. (3.1), we see that

$$q(t)y^2(1+\alpha y^2+\beta z^2)-\sin\sqrt{t+1}\,y \geqslant q(t)y^2-\sin\sqrt{t+1}\,y$$

for all $(t, y, z) \in [0, \infty) \times \mathbf{R}^2$. The zeros of the forcing term $\sin\sqrt{t+1}$ are $(k\pi)^2-1$, $k \in \mathbf{Z}$.

For any $T \geqslant 0$, choose $k \in \mathbf{N}$ sufficiently large so that $(k\pi)^2-1 \geqslant T$ and set $a_1=(k\pi)^2-1$, $b_1=(k+1)^2\pi^2-1$. Taking $u(t)=\sin\sqrt{t+1}$, it is easy to verify that $u \in D((k\pi)^2-1, (k+1)^2\pi^2-1)$ and

$$\begin{aligned}
&\int_{a_1}^{b_1}\{q(t)u^2(t)-p(t)u'^2(t)\}\,\mathrm{d}t\\
&\quad\geqslant \int_{(k\pi)^2-1}^{(k+1)^2\pi^2-1}\left\{\frac{1}{4\sqrt{t+1}}\sin^2\sqrt{t+1}-\frac{1}{4\sqrt{t+1}}\cos^2\sqrt{t+1}\right\}\mathrm{d}t\\
&\quad= \int_{k\pi}^{(k+1)\pi}\left\{\frac{1}{2}\sin^2 t-\frac{1}{2}\cos^2 t\right\}\mathrm{d}t=0.
\end{aligned}$$

Similarly, for $a_2=(k+1)^2\pi^2-1$, $b_2=(k+2)^2\pi^2-1$, we can show that

$$\int_{a_2}^{b_2}\{q(t)u^2(t)-p(t)u'^2(t)\}\,\mathrm{d}t \geqslant 0.$$

It follows from Corollary 2.4 that Eq. (3.1) is oscillatory.

The second example illustrates Corollary 2.5.

Example 3.2. Let $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $v>1$. Then the forced second-order differential equation

$$[(1+a\cos^2 t)y'(t)]'+(\beta\sin t)|y(t)|^v(1+by^2(t)+cy'^2(t))\operatorname{sgn} y(t)=\cos t, \quad t \geqslant 0 \tag{3.2}$$

is oscillatory provided

$$\beta^{1/v} \geqslant \left(1+\frac{a}{2}\right)\pi \Bigg/ \left[4v(v-1)^{(1-v)/v}\int_0^{\pi/2}[\sin^{2+1/v} t][\cos^{3-1/v} t]\,\mathrm{d}t\right]. \tag{3.3}$$

We will use Corollary 2.5 to show oscillation. For any $T \geqslant 0$, choose $k \in \mathbf{N}$ sufficiently large so that $2k\pi \geqslant T$ and set $a_1=2k\pi$, $b_1=a_2=2k\pi+\pi/2$ and $b_2=2k\pi+\pi$. Let $q(t)=\beta\sin t$ and $e(t)=\cos t$, then $q(t) \geqslant 0$ on $[a_1, b_1] \cup [a_2, b_2]$,

$$e(t)\begin{cases}\geqslant 0, & t \in [a_1, b_1],\\ \leqslant 0, & t \in [a_2, b_2],\end{cases}$$

and

$$\frac{(\beta\sin t)|y|^v(1+by^2+cz^2)\operatorname{sgn} y-\cos t}{y} \geqslant q(t)|y|^{v-1}-\frac{e(t)}{y}$$

for all $t \in [a_1, b_1] \cup [a_2, b_2]$, $y \neq 0$, $z \in \mathbf{R}$.

On the other hand, taking $u(t)=\sin 2t$ we have

$$\begin{aligned}
&v(v-1)^{(1-v)/v}\int_{a_1}^{b_1}u^2(t)q^{1/v}(t)|e(t)|^{(v-1)/v}\,\mathrm{d}t-\int_{a_1}^{b_1}p(t)u'^2(t)\,\mathrm{d}t\\
&\quad=4v(v-1)^{(1-v)/v}\beta^{1/v}\int_{2k\pi}^{2k\pi+\pi/2}[\sin^{2+1/v} t][\cos^{3-1/v} t]\,\mathrm{d}t-4\int_{2k\pi}^{2k\pi+\pi/2}\cos^2 2t\,(1+a\cos^2 t)\,\mathrm{d}t\\
&\quad=4v(v-1)^{(1-v)/v}\beta^{1/v}\int_0^{\pi/2}[\sin^{2+1/v} t][\cos^{3-1/v} t]\,\mathrm{d}t-\left(1+\frac{a}{2}\right)\pi \geqslant 0
\end{aligned}$$

and similarly,

$$v(v-1)^{(1-v)/v}\int_{a_2}^{b_2} u^2(t)q^{1/v}(t)|e(t)|^{(v-1)/v}\,\mathrm{d}t-\int_{a_2}^{b_2} p(t)u'^2(t)\,\mathrm{d}t\geqslant 0.$$

Therefore, by Corollary 2.5, we see that Eq. (3.2) is oscillatory.

Remark 3.3. When $a=b=c=0$, Eq. (3.2) reduces to the following equation as in the example of [3]:

$$y''(t)+\beta\sin t|y(t)|^v\,\mathrm{sgn}\;y(t)=\cos t,\qquad t\geqslant 0$$

and condition (3.3) becomes

$$\beta^{1/v}\geqslant\pi\Bigg/\left[4v(v-1)^{(1-v)/v}\int_0^{\pi/2}[\sin^{2+1/v}t][\cos^{3-1/v}t]\,\mathrm{d}t\right].\tag{3.4}$$

Since $v>1$, $v(v-1)^{(1-v)/v}=v^{1/v}(v/(v-1))^{(v-1)/v}>1$. Hence, our condition (3.4) is sharper than that in [3]

$$\beta^{1/v}\geqslant\pi\Bigg/\left[4\int_0^{\pi/2}[\sin^{2+1/v}t][\cos^{3-1/v}t]\,\mathrm{d}t\right].$$

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